MATH 1650: REAL ZEROS OF POLYNOMIALS

EXAMPLE: Find all real zeros of the following polynomial functions using the following procedure:

- Use the Rational Zeros Theorem (RZT) to list all of the possible rational zeros.
- Graph the function using a graphing utility.
- Use the graph to shorten the list of possible rational zeros.
- Use synthetic division to find the real zeros of the function, and state their multiplicities.

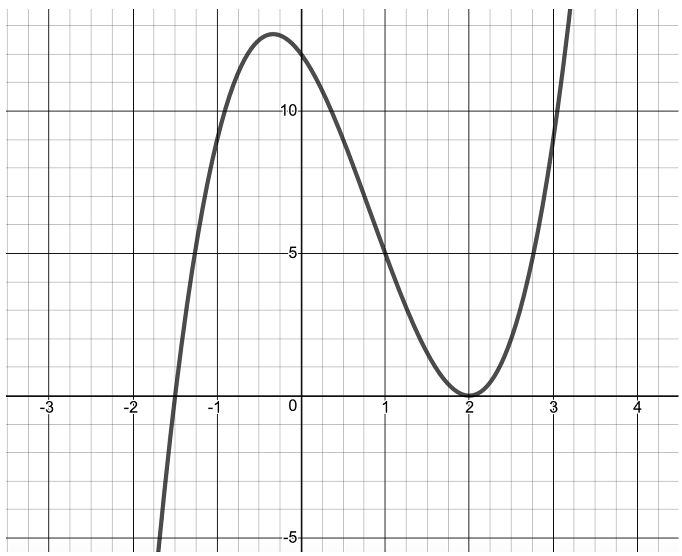
1. $p(x) = 2x^3 - 5x^2 - 4x + 12$

   - Use the Rational Zeros Theorem to list all of the possible rational zeros.

     The RZT says the rational zeros of p appear among the list $\pm\dfrac{\text{factors of } 12}{\text{factors of } 2} = \pm\dfrac{1, 2, 3, 4, 6, 12}{1, 2}$.

     We get the list: $\left\{\pm 1, \pm\dfrac{1}{2}, \pm 2, \pm 3, \pm\dfrac{3}{2}, \pm 4, \pm 6, \pm 12\right\}$

   - Graph the function using a graphing utility.



   - Use the graph to shorten the list of possible rational zeros.

     Looking at the graph, $x = 2$ looks like a good bet for a real zero. Since p is degree 3, all we need is one division to work out to get the problem reduced to solving a quadratic equation.

   - We divide $x = 2$ into $p(x)$:

$$\begin{array}{r|rrrr} 2 & 2 & -5 & -4 & 12 \\ & \downarrow & 4 & -2 & -12 \\ \hline & 2 & -1 & -6 & \boxed{0} \end{array}$$

Hence, $x = 2$ is a zero and our quotient polynomial is $2x^2 - x - 6$.

Solving $2x^2 - x - 6 = 0$ gives $(2x + 3)(x - 2) = 0$ so $x = -\dfrac{3}{2}$ or $x = 2$.

Hence, $x = 2$ is a zero of multiplicity 2 and $x = -\dfrac{3}{2}$ of multiplicity 1.

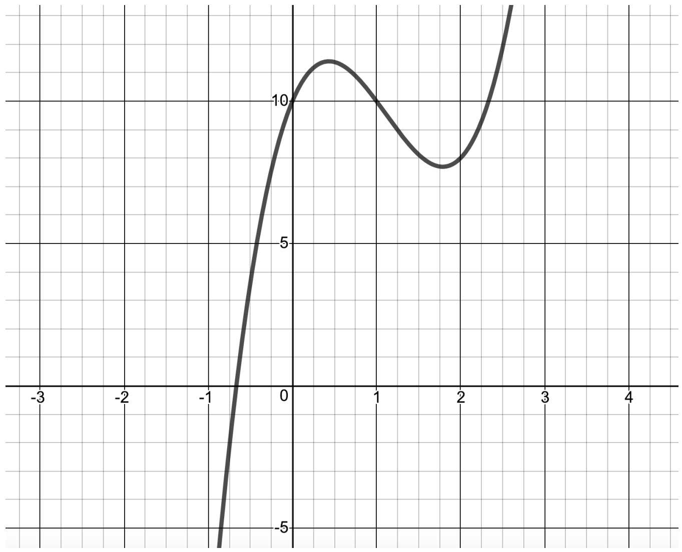
2. $q(z) = 3z^3 - 10z^2 + 7z + 10$

- Use the Rational Zeros Theorem to list all of the possible rational zeros.

  The RZT says the rational zeros of q appear among the list $\pm \dfrac{\text{factors of } 10}{\text{factors of } 3} = \pm \dfrac{1, 2, 5, 10}{1, 3}$.

  We get the list: $\left\{ \pm 1, \pm \dfrac{1}{3}, \pm 2, \pm \dfrac{2}{3}, \pm 5, \pm \dfrac{5}{3}, \pm 10, \pm \dfrac{10}{3} \right\}$

- Graph the function using a graphing utility.



- Use the graph to shorten the list of possible rational zeros.

  Looking at the graph, $z = -\dfrac{2}{3}$ looks like a good bet for a real zero. Since q is degree 3, all we need is one division to work out to get the problem reduced to solving a quadratic equation.

- We divide $z = -\dfrac{2}{3}$ into $p(x)$:

$$\begin{array}{r|rrrr} -\frac{2}{3} & 3 & -10 & 7 & 10 \\ & \downarrow & -2 & 8 & -10 \\ \hline & 3 & -12 & 15 & \boxed{0} \end{array}$$

  Hence, $z = -\dfrac{2}{3}$ is a zero and our quotient polynomial is $3z^2 - 12z + 15$.

  We factor $3z^2 - 12z + 15 = 0$ as $3(z^2 - 4z + 5) = 0$ so $z^2 - 4z + 5 = 0$.

  A quick check shows us $z^2 - 4z + 5 = 0$ doesn't factor nicely, so we try the quadratic formula :

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{16 - 20}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = \frac{2(2 \pm i)}{2} = 2 \pm i.$$

  Hence, $z = -\dfrac{2}{3}$ is the only **real** zero and has multiplicity 1.

EXAMPLE: Solve the following equations and/or inequalities.

- $4x^5 + 8x^4 + 5x^3 = 5x^2 + 4x - 2$

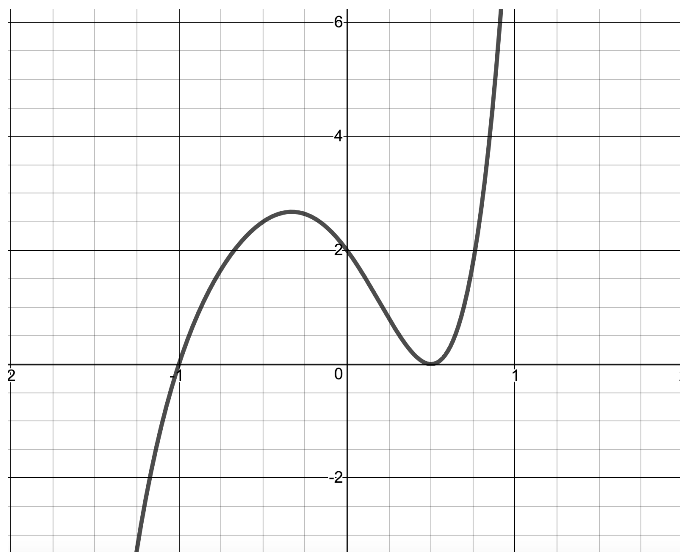
  HINT: Rewrite the equation in the form $f(x) = 0$ and find the real zeros of f.

  We rewrite the equation as: $4x^5 + 8x^4 + 5x^3 - 5x^2 - 4x + 2 = 0$.

  We let $f(x) = 4x^5 + 8x^4 + 5x^3 - 5x^2 - 4x + 2$ and set about finding the zeros of f.

  The RZT tells us the possible rational zeros are: $\pm\dfrac{\text{factors of } 2}{\text{factors of } 4}$ or $\left\{\pm 1, \pm\dfrac{1}{2}, \pm\dfrac{1}{4}, \pm 2\right\}$.

  We graph $y = f(x)$ below.



  From the graph, $x = -1$ looks good as well as $x = \dfrac{1}{2}$. Moreover, the graph appears to 'bounce' at $x = \dfrac{1}{2}$, suggesting even multiplicity.

  Since f is degree 5, we'll need **three** divisions to get down to a a quadratic factor.

  We begin with $x = -1$ and try $x = \dfrac{1}{2}$ twice:

$$
\begin{array}{r|rrrrrr}
-1 & 4 & 8 & 5 & -5 & -4 & 2 \\
 & \downarrow & -4 & -4 & -1 & 6 & -2 \\
\hline
\frac{1}{2} & 4 & 4 & 1 & -6 & 2 & \boxed{0} \\
 & \downarrow & 2 & 3 & 2 & -2 & \\
\hline
\frac{1}{2} & 4 & 6 & 4 & -4 & \boxed{0} & \\
 & \downarrow & 2 & 4 & 4 & & \\
\hline
 & 4 & 8 & 8 & \boxed{0} & &
\end{array}
$$

  Our quotient polynomial is $4x^2 + 8x + 8$. Solving $4x^2 + 8x + 8 = 0$ gives $4(x^2 + 2x + 2) = 0$, which reduces to $x^2 + 2x + 2 = 0$. This quadratic doesn't factor easily, so we use the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4 - 8}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = \frac{2(-1 \pm i)}{2} = -1 \pm i$$

  Hence, the only **real** zeros of f are $x = -1$ and $x = \dfrac{1}{2}$, which means the only real **solutions** to the original equation: $4x^5 + 8x^4 + 5x^3 = 5x^2 + 4x - 2$ are $x = -1$ and $x = \dfrac{1}{2}$.

- $x^4 + 12x^2 + 1 \geq 7x(x^2+1)$

  HINT: Rewrite the inequality in the form $f(x) \geq 0$ and make a Sign Diagram.

  We first simplify: $x^4 + 12x^2 + 1 \geq 7x(x^2+1)$ as $x^4 + 12x^2 + 1 \geq 7x^3 + 7x$.

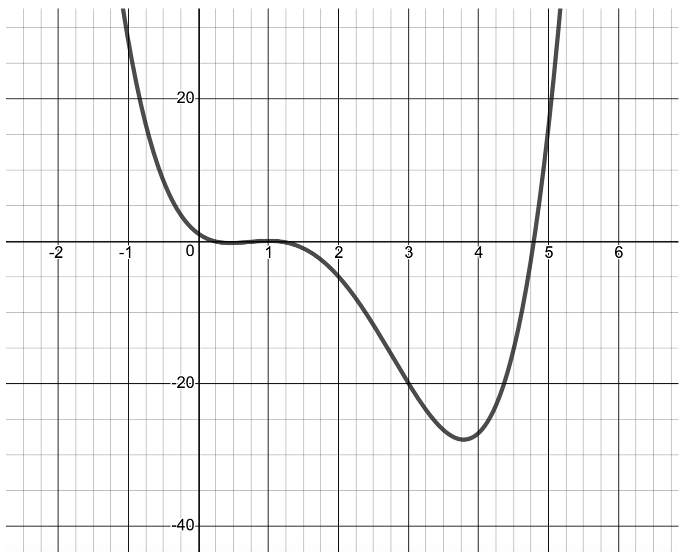
  Rewriting the inequality as $x^4 - 7x^3 + 12x^2 - 7x + 1 \geq 0$, we let $f(x) = x^4 - 7x^3 + 12x^2 - 7x + 1$.

  Solving the inequality is now equivalent to solving $f(x) \geq 0$ which we'll solve using a sign diagram.

  First, we need to solve $f(x) = 0$, that is, we need to find the zeros of $f(x) = x^4 - 7x^3 + 12x^2 - 7x + 1$.

  The RZT tells us the possible rational zeros are $\pm\dfrac{\text{factors of } 1}{\text{factors of } 1}$ which gives just $\{\pm 1\}$.

  Looking at the graph, $x = 1$ looks good and appears to have multiplicity more than 1 (maybe 3?)



  Since $f(x)$ is degree 4, we need two divisions to obtain a quadratic quotient, so we divide $x = 1$ twice:

$$\begin{array}{r|rrrrr}
1 & 1 & -7 & 12 & -7 & 1 \\
 & \downarrow & 1 & -6 & 6 & -1 \\
\hline
1 & 1 & -6 & 6 & -1 & \boxed{0} \\
 & \downarrow & 1 & -5 & 1 & \\
\hline
 & 1 & -5 & 1 & \boxed{0} &
\end{array}$$

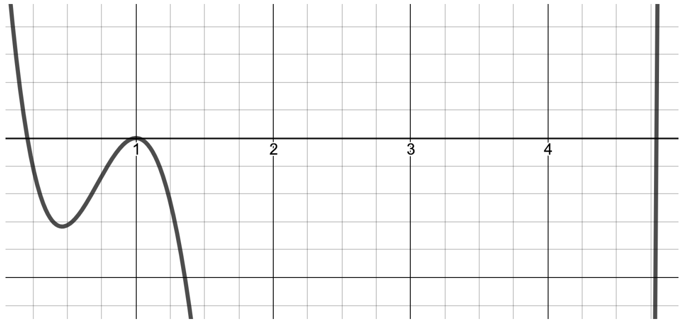
  The quotient polynomial here is $x^2 - 5x + 1$. Solving $x^2 - 5x + 1 = 0$ requires the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{5 \pm \sqrt{25-4}}{2} = \frac{5 \pm \sqrt{21}}{2}$$

  Hence, $x = 1$ is a zero of multiplicity 2 with $x = \dfrac{5-\sqrt{21}}{2}$ and $x = \dfrac{5+\sqrt{21}}{2}$ each a zero of multiplicity 1.

  Note this tells us that our original graph of $y = f(x)$ is somewhat misleading, since it suggests $x = 1$ is a zero of multiplicity 3 with only one other real zero!

Indeed, zooming in near the x-axis, we find:



We proceed to make a sign diagram:

$$\begin{array}{ccccccccc} & (+) & 0 & (-) & 0 & (-) & 0 & (+) & \\ -\infty & \longleftarrow & \vert & \text{———} & \vert & \text{———} & \vert & \longrightarrow & \infty \\ & & \dfrac{5-\sqrt{21}}{2} & & 1 & & \dfrac{5+\sqrt{21}}{2} & & \end{array}$$

Hence, $f(x) \geq 0$ on $\left(-\infty, \dfrac{5-\sqrt{21}}{2}\right] \cup \{1\} \cup \left[\dfrac{5+\sqrt{21}}{2}, \infty\right)$, which is our final answer.